

GLOBAL SOLUTION TO THE INCOMPRESSIBLE OLDROYD-B MODEL IN HYBRID BESOV SPACES

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ABSTRACT. This paper is dedicated to the Cauchy problem of the incompressible Oldroyd-B model with general coupling constant $\omega \in (0, 1)$. It is shown that this set of equations admits a unique global solution in a certain hybrid Besov spaces for small initial data in $\dot{H}^s \cap \dot{B}_{2,1}^{\frac{d}{2}}$ with $-\frac{d}{2} < s < \frac{d}{2} - 1$. In particular, if $d \geq 3$, and taking $s = 0$, then $\dot{H}^0 \cap \dot{B}_{2,1}^{\frac{d}{2}} \approx B_{2,1}^{\frac{d}{2}}$. Since $B_{2,\infty}^s \hookrightarrow B_{2,1}^{\frac{d}{2}}$, $s > \frac{d}{2}$, this result extends the work by Chen and Miao [Nonlinear Anal., 68(2008), 1928–1939].

1. INTRODUCTION

We consider a typical model for viscoelastic fluids, the so called Oldroyd-B model [26] in this paper. This type of fluids is described by the following set of equations

$$(1.1) \quad \begin{cases} u_t + (u \cdot \nabla)u - \eta_s \Delta u + \nabla \Pi = \operatorname{div} \tau, \\ \operatorname{div} u = 0, \\ \lambda(\tau_t + (u \cdot \nabla)\tau + g_\alpha(\tau, \nabla u)) + \tau = 2\eta_e D(u), \end{cases}$$

where u and τ are the velocity and symmetric tensor of constrains of the fluids, respectively. Π is the pressure which is the Lagrange multiplier for the divergence free condition. The quadratic form in $(\tau, \nabla u)$ is given by $g_\alpha(\tau, \nabla u) := \tau W(u) - W(u)\tau - \alpha(D(u)\tau + \tau D(u))$ for some $\alpha \in [-1, 1]$, and $D(u) := \frac{1}{2}(\nabla u + (\nabla u)^\top)$, $W(u) := \frac{1}{2}(\nabla u - (\nabla u)^\top)$ are the deformation tensor and the vorticity tensor, respectively. Moreover, the parameter $\eta_s := \eta\mu/\lambda$ denotes the solvent viscosity, and $\eta_e := \eta - \eta_s$ denotes the polymer viscosity, where η is the total viscosity of the fluid, $\lambda > 0$ is the relaxation time, and μ is the retardation time with $0 < \mu < \lambda$.

In the following, we would like to study system (1.1) in dimensionless variables, which takes the form

$$(1.2) \quad \begin{cases} \operatorname{Re}(u_t + (u \cdot \nabla)u) - (1 - \omega)\Delta u + \nabla \Pi = \operatorname{div} \tau, \\ \operatorname{We}(\tau_t + (u \cdot \nabla)\tau + g_\alpha(\tau, \nabla u)) + \tau = 2\omega D(u), \\ \operatorname{div} u = 0, \end{cases}$$

with parameters Reynolds number Re , Weissenberg number We and coupling constant $\omega := 1 - \frac{\mu}{\lambda} \in (0, 1)$ of the fluid. For more details of the modeling, please refer to [9, 12, 28] and references therein.

Some of the previous works in this direction can be summarized as follows. To our best knowledge, the incompressible Oldroyd-B model was firstly studied by Guillopé and Saut [11], where they obtained a unique local strong solution to system (1.2) in suitable Sobolev spaces $H^s(\Omega)$ for the situation of a *bounded domain* $\Omega \subset \mathbb{R}^3$. Moreover, this solution is global provided both the data and the coupling constant ω between the two equations are sufficiently small. The extensions to these results to the L^p -setting can be found in [10]. Similar results on *exterior domains* was established by Hieber, Naito and Shibata [13]. The well-posedness results in *scaling invariant* Besov spaces on \mathbb{R}^d , $d \geq 2$ were first given by Chemin and Masmoudi [5].

All these results above were constructed under the assumption that the coupling constant ω is small enough. This means that the coupling effect between the velocity u and the symmetric tensor

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of constrains τ is weak and hence system (1.2) corresponds closely to the classical incompressible Navier-Stokes equations. From both the physical and mathematical point of view, it is more interesting to consider the strong coupling case, for which the coupling constant ω is not small. As a matter of fact, the studies in this direction have thrown up some interesting results. For the situation of *bounded domains*, the smallness restriction on the coupling constant ω in [11] was removed by Molinet and Talhouk [25]. As for the *exterior domains*, Fang, Hieber and the author [8] improved the main result in [13] to the situation of *non-small* coupling constant. In the *whole space* \mathbb{R}^d case, Chen and Miao [6] obtained global solutions to system (1.1) with small initial data in $B_{2,\infty}^s$, $s > \frac{d}{2}$. For the critical L^p framework, the smallness restriction on ω in [5] was removed by Fang, Zhang and the author [30] very recently. Existence of global *weak solutions* for large data and strong coupling was proved by Lions and Masmoudi in [24] for the case $\alpha = 0$. The general case $\alpha \neq 0$ is still open up to now. For the Oldroyd-B fluids with *diffusive stress*, Constantin and Kliegl [7] proved the global regularity of solutions in two dimensional case.

Besides, we would like to point out that there are some other results on Oldroyd-B fluids in the literature. Indeed, Chemin and Masmoudi [5] gave some *blow-up criterions* both for 2D and 3D cases. Later on, the 2D case was improved by Lei, Masmoudi and Zhou in [20]. As for the 3D case, Kupferman, Mangoubi and Titi [15] established a Beale-Kato-Majda type blow-up criterion in terms of the $L_t^1(L_x^\infty)$ norm of τ in the *zero Reynolds number* regime. Further results, describing the *incompressible limit problems* for Oldroyd-B fluids, can be found in [12, 16] for well-prepared initial data, and in [9] for ill-prepared initial data. An approach based on *deformation tensor* was developed in [14, 17, 18, 19, 21, 22, 23, 27, 29].

The aim of this paper is to study the incompressible Oldroyd-B model (1.2) with *non-small* coupling constant ω . We establish the global solutions to system (1.2) with small data u_0 and τ_0 lying in $\dot{B}^s \approx \dot{H}^s \cap \dot{B}_{2,1}^{\frac{d}{2}}, -\frac{d}{2} < s < \frac{d}{2} - 1$. Like all the previous results [6, 8, 24, 25] in L^2 framework with *non-small* coupling constant ω , the key point of the proof is to use the cancelation relation

$$(\operatorname{div} \tau | u) + (D(u) | \tau) = 0.$$

The global estimates can be divided into two parts. For the initial data in $\dot{B}_{2,1}^{\frac{d}{2}}$, owing to the Bernstein's inequality, we can obtain both the smoothing effect of the velocity u and the damping effect of the symmetric tensor of constrains τ in the high frequency case. While in the low frequency case, the estimate fails to be true since u and τ are treated as a whole, and $\|\dot{\Delta}_q u\|_{L^2} + \|\dot{\Delta}_q \tau\|_{L^2}$ can not be dominated by $\|\nabla \dot{\Delta}_q u\|_{L^2} + \|\dot{\Delta}_q \tau\|_{L^2}$ any more (see (3.11) for details). In order to deal with the low frequency part, we impose an extra condition on the initial data. This leads to the estimates for initial data in \dot{H}^s . It is worth noting that the estimates of nonlinear terms necessitate bounding the term $\|u\|_{L_t^2(\dot{B}_{2,1}^{\frac{d}{2}})}$. To do so, we decompose $\|u\|_{L_t^2(\dot{B}_{2,1}^{\frac{d}{2}})}$ into $\|u\|_{L_t^2(\dot{B}_{2,1}^{\frac{d}{2}})}^l$ and $\|u\|_{L_t^2(\dot{B}_{2,1}^{\frac{d}{2}})}^h$. In particular, the low frequency part is bounded by $\|u\|_{L_t^2(\dot{H}^{s+1})}^l$; that is why we need $s < \frac{d}{2} - 1$. Finally, combing the two parts estimates with initial data in $\dot{B}_{2,1}^{\frac{d}{2}}$ and \dot{H}^s , we obtain the global estimates for (u, τ) .

Notations. For $s \in \mathbb{R}$, set

$$\|u\|_{\dot{B}_{2,1}^s}^l := \sum_{q < 0} 2^{qs} \|\dot{\Delta}_q u\|_{L^2}, \quad \text{and} \quad \|u\|_{\dot{B}_{2,1}^s}^h := \sum_{q \geq 0} 2^{qs} \|\dot{\Delta}_q u\|_{L^2}.$$

Further more, let us denote by \dot{B}_h^s the space which consists of functions $u \in \mathcal{S}'$, such that $\|u\|_{\dot{B}_{2,1}^s}^h < \infty$. Throughout the paper, C denotes various ‘‘harmless’’ positive constants, and we sometimes use the notation $A \lesssim B$ as an equivalent to $A \leq CB$. The notation $A \approx B$ means that $A \lesssim B$ and $B \lesssim A$.

We shall obtain the existence and uniqueness of a solution (u, τ) to (1.2) in the following space.

Definition 1.1. For $T > 0$, and $s \in \mathbb{R}$, let us denote

$$\mathcal{E}_T^s := \left(\tilde{C}_T(\dot{B}^s) \cap L_T^2(\dot{H}^{s+1}) \cap L_T^1(\dot{B}_h^{\frac{d}{2}+1}) \right)^d \times \left(\tilde{C}_T(\dot{B}^s) \cap L_T^2(\dot{H}^s) \cap L_T^1(\dot{B}_h^{\frac{d}{2}}) \right)^{d \times d}.$$

We use the notation \mathcal{E}^s if $T = \infty$, changing $[0, T]$ into $[0, \infty)$ in the definition above. The definition of space \mathcal{B}^s can be found in Section 2.

Our main result reads as follows:

Theorem 1.1. *Let $d \geq 2$, $-\frac{d}{2} < s < \frac{d}{2} - 1$. Assume that $(u_0, \tau_0) \in (\dot{\mathcal{B}}^s)^d \times (\dot{\mathcal{B}}^s)^{d \times d}$ with $\operatorname{div} u_0 = 0$. There exist two positive constants c and M , depending on s, d, ω, Re and We , such that if*

$$\|u_0\|_{\dot{\mathcal{B}}^s} + \|\tau_0\|_{\dot{\mathcal{B}}^s} \leq c,$$

system (1.2) admits a unique global solution (u, τ) in \mathcal{E}^s with

$$\|(u, \tau)\|_{\mathcal{E}^s} \leq M(\|u_0\|_{\dot{\mathcal{B}}^s} + \|\tau_0\|_{\dot{\mathcal{B}}^s}).$$

2. THE FUNCTIONAL TOOL BOX

The results of the present paper rely on the use of a dyadic partition of unity with respect to the Fourier variables, the so-called the *Littlewood-Paley decomposition*. Let us briefly explain how it may be built in the case $x \in \mathbb{R}^d$ which the readers may see more details in [1, 3]. Let (χ, φ) be a couple of C^∞ functions satisfying

$$\operatorname{Supp} \chi \subset \{|\xi| \leq \frac{4}{3}\}, \quad \operatorname{Supp} \varphi \subset \{\frac{3}{4} \leq |\xi| \leq \frac{8}{3}\},$$

and

$$\chi(\xi) + \sum_{q \geq 0} \varphi(2^{-q}\xi) = 1,$$

$$\sum_{q \in \mathbb{Z}} \varphi(2^{-q}\xi) = 1, \quad \text{for } \xi \neq 0.$$

Set $\varphi_q(\xi) = \varphi(2^{-q}\xi)$, $h_q = \mathcal{F}^{-1}(\varphi_q)$, and $\tilde{h} = \mathcal{F}^{-1}(\chi)$. The dyadic blocks and the low-frequency cutoff operators are defined for all $q \in \mathbb{Z}$ by

$$\dot{\Delta}_q u = \varphi(2^{-q}\mathbf{D})u = \int_{\mathbb{R}^d} h_q(y)u(x-y)dy,$$

$$\dot{S}_q u = \chi(2^{-q}\mathbf{D})u = \int_{\mathbb{R}^d} \tilde{h}_q(y)u(x-y)dy.$$

Then

$$(2.1) \quad u = \sum_{q \in \mathbb{Z}} \dot{\Delta}_q u,$$

holds for tempered distributions *modulo polynomials*. As working modulo polynomials is not appropriate for nonlinear problems, we shall restrict our attention to the set \mathcal{S}'_h of tempered distributions u such that

$$\lim_{q \rightarrow -\infty} \|\dot{S}_q u\|_{L^\infty} = 0.$$

Note that (2.1) holds true whenever u is in \mathcal{S}'_h and that one may write

$$\dot{S}_q u = \sum_{p \leq q-1} \dot{\Delta}_p u.$$

Besides, we would like to mention that the Littlewood-Paley decomposition has a nice property of quasi-orthogonality:

$$(2.2) \quad \dot{\Delta}_p \dot{\Delta}_q u \equiv 0 \text{ if } |p - q| \geq 2 \text{ and } \dot{\Delta}_p (\dot{S}_{q-1} u \dot{\Delta}_q u) \equiv 0 \text{ if } |p - q| \geq 5.$$

One can now give the definition of homogeneous Besov spaces.

Definition 2.1. For $s \in \mathbb{R}$, $(p, r) \in [1, \infty]^2$, and $u \in \mathcal{S}'(\mathbb{R}^d)$, we set

$$\|u\|_{\dot{B}_{p,r}^s} = \|2^{sq} \dot{\Delta}_q u\|_{L^p} \|_{\ell^r}.$$

We then define the space $\dot{B}_{p,r}^s := \{u \in \mathcal{S}'_h(\mathbb{R}^d), \|u\|_{\dot{B}_{p,r}^s} < \infty\}$.

Remark 2.1. The inhomogeneous Besov spaces can be defined in a similar way. Indeed, for $u \in \mathcal{S}'(\mathbb{R}^d)$, we set

$$\begin{aligned} \Delta_q u &= 0 \text{ if } q < -1, \quad \Delta_{-1} u = \chi(D)u, \\ \Delta_q u &= \varphi(2^{-q}D)u \text{ if } q \geq 0, \quad \text{and} \quad S_q u = \sum_{p \leq q-1} \Delta_p u. \end{aligned}$$

Then for all $u \in \mathcal{S}'(\mathbb{R}^d)$, we have the inhomogeneous Littlewood-Paley decomposition $u = \sum_{q \in \mathbb{Z}} \Delta_q u$, and for $(p, r) \in [1, +\infty]^2$, $s \in \mathbb{R}$, we define the inhomogeneous Besov space $B_{p,r}^s$ as

$$B_{p,r}^s = \{u \in \mathcal{S}'(\mathbb{R}^d), \|u\|_{B_{p,r}^s} := \|2^{sq} \|\Delta_q u\|_{L^p}\|_{\ell^r} < \infty\}$$

We also need the following hybrid Besov space in this paper.

Definition 2.2. For $s \in \mathbb{R}$, and $u \in \mathcal{S}'(\mathbb{R}^d)$, we set

$$\|u\|_{\dot{\mathcal{B}}^s} = \left(\sum_{q < 0} 2^{2qs} \|\dot{\Delta}_q u\|_{L^2}^2 \right)^{\frac{1}{2}} + \sum_{q \geq 0} 2^{q\frac{d}{2}} \|\dot{\Delta}_q u\|_{L^2}.$$

We then define the space $\dot{\mathcal{B}}^s := \{u \in \mathcal{S}'_h(\mathbb{R}^d), \|u\|_{\dot{\mathcal{B}}^s} < \infty\}$.

Remark 2.2. It is easy to verify that

$$\dot{\mathcal{B}}^s \approx \dot{H}^s \cap \dot{B}_{2,1}^{\frac{d}{2}},$$

provided $s < \frac{d}{2}$.

The following lemma describes the way derivatives act on spectrally localized functions.

Lemma 2.1 (Bernstein's inequalities). *Let $k \in \mathbb{N}$ and $0 < r < R$. There exists a constant C depending on r, R and d such that for all $(a, b) \in [1, \infty]^2$, we have for all $\lambda > 0$ and multi-index α*

- If $\text{Supp } \hat{f} \subset B(0, \lambda R)$, then $\sup_{|\alpha|=k} \|\partial^\alpha f\|_{L^b} \leq C^{k+1} \lambda^{k+d(\frac{1}{a}-\frac{1}{b})} \|f\|_{L^a}$.
- If $\text{Supp } \hat{f} \subset \mathbb{C}(0, \lambda r, \lambda R)$, then $C^{-k-1} \lambda^k \|f\|_{L^a} \leq \sup_{|\alpha|=k} \|\partial^\alpha f\|_{L^a} \leq C^{k+1} \lambda^k \|f\|_{L^a}$.

Next we recall a few nonlinear estimates in Besov spaces which may be obtained by means of paradifferential calculus. Firstly introduced by J. M. Bony in [2], the paraproduct between f and g is defined by

$$\dot{T}_f g = \sum_{q \in \mathbb{Z}} \dot{S}_{q-1} f \dot{\Delta}_q g,$$

and the remainder is given by

$$\dot{R}(f, g) = \sum_{q \geq -1} \dot{\Delta}_q f \tilde{\Delta}_q g$$

with

$$\tilde{\Delta}_q g = (\dot{\Delta}_{q-1} + \dot{\Delta}_q + \dot{\Delta}_{q+1})g.$$

We have the following so-called Bony's decomposition:

$$(2.3) \quad fg = \dot{T}_f g + \dot{T}_g f + \dot{R}(f, g).$$

The paraproduct \dot{T} and the remainder \dot{R} operators satisfy the following continuous properties.

Proposition 2.1. *For all $s \in \mathbb{R}$, $\sigma > 0$, and $1 \leq p, p_1, p_2, r, r_1, r_2 \leq \infty$, the paraproduct \dot{T} is a bilinear, continuous operator from $L^\infty \times \dot{B}_{p,r}^s$ to $\dot{B}_{p,r}^s$ and from $\dot{B}_{\infty,r_1}^{-\sigma} \times \dot{B}_{p,r_2}^s$ to $\dot{B}_{p,r}^{s-\sigma}$ with $\frac{1}{r} = \min\{1, \frac{1}{r_1} + \frac{1}{r_2}\}$. The remainder \dot{R} is bilinear continuous from $\dot{B}_{p_1,r_1}^{s_1} \times \dot{B}_{p_2,r_2}^{s_2}$ to $\dot{B}_{p,r}^{s_1+s_2}$ with $s_1 + s_2 > 0$, $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} \leq 1$, and $\frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2} \leq 1$.*

In view of (2.3), Proposition 2.1 and Bernstein's inequalities, one easily deduces the following product estimates:

Corollary 2.1. *There hold:*

$$(2.4) \quad \|uv\|_{\dot{H}^s} \leq C \|u\|_{\dot{B}_{2,1}^{\frac{d}{2}}}^{\frac{d}{2}} \|v\|_{\dot{H}^s}, \quad \text{if } s \in \left(-\frac{d}{2}, \frac{d}{2}\right).$$

$$(2.5) \quad \|uv\|_{\dot{H}^s} \leq C \|u\|_{\dot{H}^{s+1}} \|v\|_{\dot{B}_{2,\infty}^{\frac{d}{2}-1}}, \quad \text{if } s \in \left(-\frac{d}{2}, \frac{d}{2} - 1\right).$$

and

$$(2.6) \quad \|uv\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \leq C \|u\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \|v\|_{\dot{B}_{2,1}^{\frac{d}{2}}}.$$

The study of non-stationary PDEs requires spaces of the type $L_T^\rho(X) = L^\rho(0, T; X)$ for appropriate Banach spaces X . In our case, we expect X to be a Besov space, so that it is natural to localize the equations through Littlewood-Paley decomposition. We then get estimates for each dyadic block and perform integration in time. But, in doing so, we obtain the bounds in spaces which are not of the type $L^\rho(0, T; \dot{B}_{p,r}^s)$. That naturally leads to the following definition introduced by Chemin and Lerner in [4].

Definition 2.3. For $\rho \in [1, +\infty]$, $s \in \mathbb{R}$, and $T \in (0, +\infty)$, we set

$$\|u\|_{\tilde{L}_T^\rho(\dot{B}_{p,r}^s)} = \left\| 2^{qs} \|\dot{\Delta}_q u(t)\|_{L_T^\rho(L^p)} \right\|_{\ell^r}$$

and denote by $\tilde{L}_T^\rho(\dot{B}_{p,r}^s)$ the subset of distributions $u \in \mathcal{D}'([0, T]; \mathcal{S}'_h(\mathbb{R}^N))$ with finite $\|u\|_{\tilde{L}_T^\rho(\dot{B}_{p,r}^s)}$ norm. When $T = +\infty$, the index T is omitted. We further denote $\tilde{C}_T(\dot{B}_{p,r}^s) = C([0, T]; \dot{B}_{p,r}^s) \cap \tilde{L}_T^\infty(\dot{B}_{p,r}^s)$.

Remark 2.3. *All the properties of continuity for the paraproduct, remainder, and product remain true for the Chemin-Lerner spaces. The exponent ρ just has to behave according to Hölder's inequality for the time variable.*

Remark 2.4. *The spaces $\tilde{L}_T^\rho(\dot{B}_{p,r}^s)$ can be linked with the classical space $L_T^\rho(\dot{B}_{p,r}^s)$ via the Minkowski inequality:*

$$\|u\|_{\tilde{L}_T^\rho(\dot{B}_{p,r}^s)} \leq \|u\|_{L_T^\rho(\dot{B}_{p,r}^s)} \quad \text{if } r \geq \rho, \quad \|u\|_{\tilde{L}_T^\rho(\dot{B}_{p,r}^s)} \geq \|u\|_{L_T^\rho(\dot{B}_{p,r}^s)} \quad \text{if } r \leq \rho.$$

3. GLOBAL EXISTENCE

In order to construct the global solutions to the incompressible Oldroyd-B model (1.2), we shall use the classical Friedrichs method to approximate the system (1.2) by a cut-off in the frequency space. Noting that this method has been applied to Oldroyd-B model in [5, 6] before, to avoid unnecessary repetition, we omit the details of approximation in this paper. In the following, the global estimates of (u, τ) will be given directly. To begin with, let us first of all localize the system (1.2) as follows,

$$(3.1) \quad \begin{cases} 2\omega \operatorname{Re} \dot{\Delta}_q u_t - 2\omega(1 - \omega) \dot{\Delta}_q \Delta u + 2\omega \nabla \dot{\Delta}_q \Pi = 2\omega \operatorname{div} \dot{\Delta}_q \tau - 2\omega \operatorname{Re} \dot{\Delta}_q (u \cdot \nabla u), \\ \operatorname{We} (\dot{\Delta}_q \tau_t + u \cdot \nabla \dot{\Delta}_q \tau) + \dot{\Delta}_q \tau = 2\omega D(\dot{\Delta}_q u) - \operatorname{We} ([\dot{\Delta}_q, u] \cdot \nabla \tau + \dot{\Delta}_q g_\alpha(\tau, \nabla u)). \end{cases}$$

Taking the L^2 inner product of (3.1)₁ and (3.1)₂ with $\dot{\Delta}_q u$ and $\dot{\Delta}_q \tau$, respectively, using the relation $(\operatorname{div} \tau | u) + (D(u) | \tau) = 0$ and the divergence free condition of u , we obtain

$$(3.2) \quad \begin{aligned} & \frac{1}{2} \frac{d}{dt} \left(2\omega \operatorname{Re} \|\dot{\Delta}_q u\|_{L^2}^2 + \operatorname{We} \|\dot{\Delta}_q \tau\|_{L^2}^2 \right) + 2\omega(1 - \omega) \|\nabla \dot{\Delta}_q u\|_{L^2}^2 + \|\dot{\Delta}_q \tau\|_{L^2}^2 \\ & \leq 2\omega \operatorname{Re} \|\dot{\Delta}_q (u \cdot \nabla u)\|_{L^2} \|\dot{\Delta}_q u\|_{L^2} + \operatorname{We} \left(\|[\dot{\Delta}_q, u] \cdot \nabla \tau\|_{L^2} + \|\dot{\Delta}_q g_\alpha(\tau, \nabla u)\|_{L^2} \right) \|\dot{\Delta}_q \tau\|_{L^2}. \end{aligned}$$

It follows that

$$\begin{aligned} & \omega \operatorname{Re} \|u\|_{\tilde{L}_t^\infty(\dot{H}^s)}^2 + \frac{\operatorname{We}}{2} \|\tau\|_{\tilde{L}_t^\infty(\dot{H}^s)}^2 + 2\omega(1 - \omega) \|\nabla u\|_{L_t^2(\dot{H}^s)}^2 + \|\tau\|_{L_t^2(\dot{H}^s)}^2 \\ & \leq \omega \operatorname{Re} \|u_0\|_{\dot{H}^s}^2 + \frac{\operatorname{We}}{2} \|\tau_0\|_{\dot{H}^s}^2 + 2\omega \operatorname{Re} \int_0^t \sum_{q \in \mathbb{Z}} 2^{2qs} \|\dot{\Delta}_q (u \cdot \nabla u)\|_{L^2} \|\dot{\Delta}_q u\|_{L^2} dt' \end{aligned}$$

$$(3.3) \quad + \text{We} \int_0^t \sum_{q \in \mathbb{Z}} 2^{2qs} \left(\|[\dot{\Delta}_q, u] \cdot \nabla \tau\|_{L^2} + \|\dot{\Delta}_q g_\alpha(\tau, \nabla u)\|_{L^2} \right) \|\dot{\Delta}_q \tau\|_{L^2} dt'.$$

Now we estimate the last three terms on the righthand side of (3.3) one by one. In fact, in view of Hölder's inequality and the product estimate (2.4), we infer that for $-\frac{d}{2} < s < \frac{d}{2}$, there holds

$$(3.4) \quad \begin{aligned} & 2\omega \text{Re} \int_0^t \sum_{q \in \mathbb{Z}} 2^{2qs} \|\dot{\Delta}_q(u \cdot \nabla u)\|_{L^2} \|\dot{\Delta}_q u\|_{L^2} dt' \\ & \leq 2\omega \text{Re} \int_0^t \|u \cdot \nabla u\|_{\dot{H}^s} \|u\|_{\dot{H}^s} dt' \\ & \leq C\omega \text{Re} \int_0^t \|\nabla u\|_{\dot{H}^s} \|u\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \|u\|_{\dot{H}^s} dt' \\ & \leq C\omega \text{Re} \|u\|_{L_t^\infty(\dot{H}^s)} \|\nabla u\|_{L_t^2(\dot{H}^s)} \|u\|_{L_t^2(\dot{B}_{2,1}^{\frac{d}{2}})}. \end{aligned}$$

Noting that if $s < \frac{d}{2} - 1$, we can bound $\|u\|_{L_t^2(\dot{B}_{2,1}^{\frac{d}{2}})}$ as follows:

$$(3.5) \quad \begin{aligned} \|u\|_{L_t^2(\dot{B}_{2,1}^{\frac{d}{2}})} & \leq \|u\|_{\tilde{L}_t^2(\dot{B}_{2,1}^{\frac{d}{2}})}^l + \|u\|_{\tilde{L}_t^2(\dot{B}_{2,1}^{\frac{d}{2}})}^h \\ & \leq C \left(\|u\|_{L_t^2(\dot{H}^{s+1})}^l + \|u\|_{\tilde{L}_t^2(\dot{B}_{2,1}^{\frac{d}{2}+\frac{1}{2}})}^h \right) \\ & \leq C \left(\|u\|_{L_t^2(\dot{H}^{s+1})} + \left(\|u\|_{L_t^\infty(\dot{B}_{2,1}^{\frac{d}{2}})}^h \|u\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}+1})}^h \right)^{\frac{1}{2}} \right) \end{aligned}$$

Inserting (3.5) into (3.4), we arrive at

$$(3.6) \quad \begin{aligned} & 2\omega \text{Re} \int_0^t \sum_{q \in \mathbb{Z}} 2^{2qs} \|\dot{\Delta}_q(u \cdot \nabla u)\|_{L^2} \|\dot{\Delta}_q u\|_{L^2} dt' \\ & \leq C\omega \text{Re} \|u\|_{L_t^\infty(\dot{H}^s)} \|\nabla u\|_{L_t^2(\dot{H}^s)} \left(\|u\|_{L_t^2(\dot{H}^{s+1})} + \left(\|u\|_{L_t^\infty(\dot{B}_{2,1}^{\frac{d}{2}})}^h \|u\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}+1})}^h \right)^{\frac{1}{2}} \right), \end{aligned}$$

with $-\frac{d}{2} < s < \frac{d}{2} - 1$. Similarly, we have

$$(3.7) \quad \begin{aligned} & \text{We} \int_0^t \sum_{q \in \mathbb{Z}} 2^{2qs} \|\dot{\Delta}_q g_\alpha(\tau, \nabla u)\|_{L^2} \|\dot{\Delta}_q \tau\|_{L^2} dt' \\ & \leq \text{We} \int_0^t \|g_\alpha(\tau, \nabla u)\|_{\dot{H}^s} \|\tau\|_{\dot{H}^s} dt' \\ & \leq C\text{We} \int_0^t \|\tau\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \|\nabla u\|_{\dot{H}^s} \|\tau\|_{\dot{H}^s} dt' \\ & \leq C\text{We} \|\tau\|_{\tilde{L}_t^\infty(\dot{B}_{2,1}^{\frac{d}{2}})} \|\nabla u\|_{L_t^2(\dot{H}^s)} \|\tau\|_{L_t^2(\dot{H}^s)}, \quad \text{for } -\frac{d}{2} < s < \frac{d}{2}. \end{aligned}$$

Finally, using Hölder's inequality and the commutator estimate, c. f. [1], for $-\frac{d}{2} - 1 < s < \frac{d}{2}$, we are led to

$$\text{We} \int_0^t \sum_{q \in \mathbb{Z}} 2^{2qs} \|[\dot{\Delta}_q, u] \cdot \nabla \tau\|_{L^2} \|\dot{\Delta}_q \tau\|_{L^2} dt'$$

$$\begin{aligned}
& \leq \text{We} \int_0^t \left(\sum_{q \in \mathbb{Z}} 2^{2qs} \|[\dot{\Delta}_q, u] \cdot \nabla \tau\|_{L^2}^2 \right)^{\frac{1}{2}} \|\tau\|_{\dot{H}^s} dt' \\
& \leq C \text{We} \int_0^t \|\nabla u\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \|\tau\|_{\dot{H}^s}^2 dt' \\
& = C \text{We} \int_0^t \|\nabla u\|_{\dot{B}_{2,1}^{\frac{d}{2}}}^l \|\tau\|_{\dot{H}^s}^2 + \|\nabla u\|_{\dot{B}_{2,1}^{\frac{d}{2}}}^h \|\tau\|_{\dot{H}^s}^2 dt' \\
& \leq C \text{We} \|\nabla u\|_{L_t^l(\dot{B}_{2,1}^{\frac{d}{2}})}^l \|\tau\|_{L_t^\infty(\dot{H}^s)} \|\tau\|_{L_t^2(\dot{H}^s)} + C \text{We} \|\nabla u\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}})}^h \|\tau\|_{L_t^\infty(\dot{H}^s)}^2 \\
(3.8) \quad & \leq C \text{We} \|\nabla u\|_{L_t^2(\dot{H}^s)} \|\tau\|_{L_t^\infty(\dot{H}^s)} \|\tau\|_{L_t^2(\dot{H}^s)} + C \text{We} \|\nabla u\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}})}^h \|\tau\|_{L_t^\infty(\dot{H}^s)}^2.
\end{aligned}$$

Substituting (3.6)–(3.8) into (3.3) yields

$$\begin{aligned}
& \omega \text{Re} \|u\|_{L_t^\infty(\dot{H}^s)}^2 + \frac{\text{We}}{2} \|\tau\|_{L_t^\infty(\dot{H}^s)}^2 + 2\omega(1-\omega) \|\nabla u\|_{L_t^2(\dot{H}^s)}^2 + \|\tau\|_{L_t^2(\dot{H}^s)}^2 \\
& \leq \omega \text{Re} \|u_0\|_{\dot{H}^s}^2 + \frac{\text{We}}{2} \|\tau_0\|_{\dot{H}^s}^2 + C \text{We} \|\tau\|_{L_t^\infty(\dot{B}_{2,1}^{\frac{d}{2}})} \|\nabla u\|_{L_t^2(\dot{H}^s)} \|\tau\|_{L_t^2(\dot{H}^s)} \\
& \quad + C \text{We} \|\nabla u\|_{L_t^2(\dot{H}^s)} \|\tau\|_{L_t^\infty(\dot{H}^s)} \|\tau\|_{L_t^2(\dot{H}^s)} + C \text{We} \|\nabla u\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}})}^h \|\tau\|_{L_t^\infty(\dot{H}^s)}^2 \\
(3.9) \quad & \quad + C \omega \text{Re} \|u\|_{L_t^\infty(\dot{H}^s)} \|\nabla u\|_{L_t^2(\dot{H}^s)} \left(\|u\|_{L_t^2(\dot{H}^{s+1})} + \left(\|u\|_{L_t^\infty(\dot{B}_{2,1}^{\frac{d}{2}})}^h \|u\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}+1})}^h \right)^{\frac{1}{2}} \right).
\end{aligned}$$

To close (3.9), we have to estimate the high frequency part of u and τ in the space $\dot{B}_{2,1}^{\frac{d}{2}}$. To this end, we first notice that

$$(3.10) \quad 2\omega \text{Re} \|\dot{\Delta}_q u\|_{L^2}^2 + \text{We} \|\dot{\Delta}_q \tau\|_{L^2}^2 \approx \left(\sqrt{2\omega \text{Re}} \|\dot{\Delta}_q u\|_{L^2} + \sqrt{\text{We}} \|\dot{\Delta}_q \tau\|_{L^2} \right)^2,$$

and for $q \geq 0$,

$$\begin{aligned}
2\omega(1-\omega) \|\nabla \dot{\Delta}_q u\|_{L^2}^2 + \|\dot{\Delta}_q \tau\|_{L^2}^2 & \approx 2\omega(1-\omega) 2^{2q} \|\dot{\Delta}_q u\|_{L^2}^2 + \|\dot{\Delta}_q \tau\|_{L^2}^2 \\
& \approx \left(\sqrt{2\omega(1-\omega)} 2^q \|\dot{\Delta}_q u\|_{L^2} + \|\dot{\Delta}_q \tau\|_{L^2} \right)^2 \\
& \geq \min \left\{ \sqrt{\frac{1-\omega}{\text{Re}}}, \frac{1}{\sqrt{\text{We}}} \right\} \left(\sqrt{2\omega(1-\omega)} 2^q \|\dot{\Delta}_q u\|_{L^2} + \|\dot{\Delta}_q \tau\|_{L^2} \right) \\
(3.11) \quad & \quad \times \left(\sqrt{2\omega \text{Re}} \|\dot{\Delta}_q u\|_{L^2} + \sqrt{\text{We}} \|\dot{\Delta}_q \tau\|_{L^2} \right).
\end{aligned}$$

It follows from (3.2), (3.10) and (3.11) that, if $q \geq 0$, there holds

$$\begin{aligned}
& \frac{d}{dt} \sqrt{2\omega \text{Re} \|\dot{\Delta}_q u\|_{L^2}^2 + \text{We} \|\dot{\Delta}_q \tau\|_{L^2}^2} \\
& \quad + \min \left\{ \sqrt{\frac{1-\omega}{\text{Re}}}, \frac{1}{\sqrt{\text{We}}} \right\} \left(\sqrt{2\omega(1-\omega)} 2^q \|\dot{\Delta}_q u\|_{L^2} + \|\dot{\Delta}_q \tau\|_{L^2} \right) \\
(3.12) \quad & \lesssim \sqrt{2\omega \text{Re}} \|\dot{\Delta}_q (u \cdot \nabla u)\|_{L^2} + \sqrt{\text{We}} \left(\|[\dot{\Delta}_q, u] \cdot \nabla \tau\|_{L^2} + \|\dot{\Delta}_q g_\alpha(\tau, \nabla u)\|_{L^2} \right).
\end{aligned}$$

Integrating the above equation with respect to t , multiplying the resulting inequality by $2^{q\frac{d}{2}}$, and then summing w. r. t. q over all the nonnegative integers, we find that

$$\sqrt{2\omega \text{Re}} \|u\|_{L_t^\infty(\dot{B}_{2,1}^{\frac{d}{2}})}^h + \sqrt{\text{We}} \|\tau\|_{L_t^\infty(\dot{B}_{2,1}^{\frac{d}{2}})}^h$$

$$\begin{aligned}
& + \min \left\{ \sqrt{\frac{1-\omega}{\text{Re}}}, \frac{1}{\sqrt{\text{We}}} \right\} \left(\sqrt{2\omega(1-\omega)} \|u\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}+1})}^h + \|\tau\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}})}^h \right) \\
& \lesssim \sqrt{2\omega \text{Re}} \|u_0\|_{\dot{B}_{2,1}^{\frac{d}{2}}}^h + \sqrt{\text{We}} \|\tau_0\|_{\dot{B}_{2,1}^{\frac{d}{2}}}^h + \sqrt{2\omega \text{Re}} \int_0^t \|u \cdot \nabla u\|_{\dot{B}_{2,1}^{\frac{d}{2}}}^h dt' \\
(3.13) \quad & + \sqrt{\text{We}} \int_0^t \sum_{q \in \mathbb{Z}} 2^{q\frac{d}{2}} \|[\Delta_q, u] \cdot \nabla \tau\|_{L^2} + \|g_\alpha(\tau, \nabla u)\|_{\dot{B}_{2,1}^{\frac{d}{2}}}^h dt'.
\end{aligned}$$

Product estimate (2.6) in Besov space and (3.5) imply that, for $-\frac{d}{2} < s < \frac{d}{2} - 1$,

$$\begin{aligned}
& \sqrt{2\omega \text{Re}} \int_0^t \|u \cdot \nabla u\|_{\dot{B}_{2,1}^{\frac{d}{2}}}^h dt' \\
& \leq C \sqrt{2\omega \text{Re}} \int_0^t \|u\|_{\dot{B}_{2,1}^{\frac{d}{2}}}^h \|\nabla u\|_{\dot{B}_{2,1}^{\frac{d}{2}}}^h dt' \\
& = C \sqrt{2\omega \text{Re}} \int_0^t \|u\|_{\dot{B}_{2,1}^{\frac{d}{2}}}^h \left(\|\nabla u\|_{\dot{B}_{2,1}^{\frac{d}{2}}}^l + \|\nabla u\|_{\dot{B}_{2,1}^{\frac{d}{2}}}^h \right) dt' \\
& \leq C \sqrt{2\omega \text{Re}} \left(\|u\|_{L_t^\infty(\dot{B}_{2,1}^{\frac{d}{2}})} \|\nabla u\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}})}^h + \|u\|_{L_t^2(\dot{B}_{2,1}^{\frac{d}{2}})} \|\nabla u\|_{L_t^2(\dot{B}_{2,1}^{\frac{d}{2}})}^l \right) \\
(3.14) \quad & \leq C \sqrt{2\omega \text{Re}} \left(\|u\|_{L_t^\infty(\dot{B}_{2,1}^{\frac{d}{2}})} \|\nabla u\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}})}^h + \left(\|u\|_{L_t^2(\dot{H}^{s+1})} + \left(\|u\|_{L_t^\infty(\dot{B}_{2,1}^{\frac{d}{2}})}^h \|u\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}+1})}^h \right)^{\frac{1}{2}} \right) \|\nabla u\|_{L_t^2(\dot{H}^s)} \right).
\end{aligned}$$

Using commutator estimate and product estimate (2.6) in Besov space again, we get for $-\frac{d}{2} < s < \frac{d}{2}$,

$$\begin{aligned}
& \sqrt{\text{We}} \int_0^t \sum_{q \in \mathbb{Z}} 2^{q\frac{d}{2}} \|[\Delta_q, u] \cdot \nabla \tau\|_{L^2} + \|g_\alpha(\tau, \nabla u)\|_{\dot{B}_{2,1}^{\frac{d}{2}}}^h dt' \\
& \leq C \sqrt{\text{We}} \int_0^t \|\nabla u\|_{\dot{B}_{2,1}^{\frac{d}{2}}}^h \|\tau\|_{\dot{B}_{2,1}^{\frac{d}{2}}}^h dt' \\
& \leq C \sqrt{\text{We}} \left(\|\tau\|_{L_t^\infty(\dot{B}_{2,1}^{\frac{d}{2}})} \|\nabla u\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}})}^h + \|\tau\|_{L_t^2(\dot{B}_{2,1}^{\frac{d}{2}})} \|\nabla u\|_{L_t^2(\dot{B}_{2,1}^{\frac{d}{2}})}^l \right) \\
(3.15) \quad & \leq C \sqrt{\text{We}} \left(\|\tau\|_{L_t^\infty(\dot{B}_{2,1}^{\frac{d}{2}})} \|\nabla u\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}})}^h + \left(\|\tau\|_{L_t^2(\dot{H}^s)} + \left(\|\tau\|_{L_t^\infty(\dot{B}_{2,1}^{\frac{d}{2}})}^h \|\tau\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}+1})}^h \right)^{\frac{1}{2}} \right) \|\nabla u\|_{L_t^2(\dot{H}^s)} \right).
\end{aligned}$$

Substituting (3.14) and (3.15) into (3.13) yields

$$\begin{aligned}
& \sqrt{2\omega \text{Re}} \|u\|_{\tilde{L}_t^\infty(\dot{B}_{2,1}^{\frac{d}{2}})}^h + \sqrt{\text{We}} \|\tau\|_{\tilde{L}_t^\infty(\dot{B}_{2,1}^{\frac{d}{2}})}^h \\
& + \min \left\{ \sqrt{\frac{1-\omega}{\text{Re}}}, \frac{1}{\sqrt{\text{We}}} \right\} \left(\sqrt{2\omega(1-\omega)} \|u\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}+1})}^h + \|\tau\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}})}^h \right) \\
& \leq \sqrt{2\omega \text{Re}} \|u_0\|_{\dot{B}_{2,1}^{\frac{d}{2}}}^h + \sqrt{\text{We}} \|\tau_0\|_{\dot{B}_{2,1}^{\frac{d}{2}}}^h \\
& + C \sqrt{2\omega \text{Re}} \left(\|u\|_{L_t^\infty(\dot{B}_{2,1}^{\frac{d}{2}})} \|\nabla u\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}})}^h + \left(\|u\|_{L_t^2(\dot{H}^{s+1})} + \left(\|u\|_{L_t^\infty(\dot{B}_{2,1}^{\frac{d}{2}})}^h \|u\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}+1})}^h \right)^{\frac{1}{2}} \right) \|\nabla u\|_{L_t^2(\dot{H}^s)} \right)
\end{aligned}$$

$$(3.16) \quad +C \sqrt{\text{We}} \left(\|\tau\|_{L_t^\infty(\dot{B}_{2,1}^{\frac{d}{2}})} \|\nabla u\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}})}^h + \left(\|\tau\|_{L_t^2(\dot{H}^s)} + \left(\|\tau\|_{L_t^\infty(\dot{B}_{2,1}^{\frac{d}{2}})} \|\tau\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}})}^h \right)^{\frac{1}{2}} \right) \|\nabla u\|_{L_t^2(\dot{H}^s)} \right).$$

Let us define

$$\begin{aligned} E_1(t) &:= \sqrt{\omega \text{Re}} \|u\|_{\tilde{L}_t^\infty(\dot{H}^s)} + \sqrt{\text{We}} \|\tau\|_{\tilde{L}_t^\infty(\dot{H}^s)} + \sqrt{\omega(1-\omega)} \|\nabla u\|_{L_t^2(\dot{H}^s)} + \|\tau\|_{L_t^2(\dot{H}^s)}, \\ E_2(t) &:= \sqrt{\omega \text{Re}} \|u\|_{\tilde{L}_t^\infty(\dot{B}_{2,1}^{\frac{d}{2}})}^h + \sqrt{\text{We}} \|\tau\|_{\tilde{L}_t^\infty(\dot{B}_{2,1}^{\frac{d}{2}})}^h + \sqrt{\omega(1-\omega)} \|\nabla u\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}})}^h + \|\tau\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}})}^h, \\ E_1(0) &:= \sqrt{\omega \text{Re}} \|u_0\|_{\dot{H}^s} + \sqrt{\text{We}} \|\tau_0\|_{\dot{H}^s}, \quad E_2(0) := \sqrt{\omega \text{Re}} \|u_0\|_{\dot{B}_{2,1}^{\frac{d}{2}}}^h + \sqrt{\text{We}} \|\tau_0\|_{\dot{B}_{2,1}^{\frac{d}{2}}}^h, \end{aligned}$$

and

$$E(t) := E_1(t) + E_2(t), \quad E(0) := E_1(0) + E_2(0).$$

Moreover, we denote

$$\begin{aligned} \kappa_1 &:= \max \left\{ \left(\frac{\text{We}}{\omega(1-\omega)} \right)^{\frac{1}{4}}, \frac{1}{(\omega(1-\omega))^{\frac{1}{4}}}, \frac{(\omega \text{Re})^{\frac{1}{4}}}{\sqrt{\omega(1-\omega)}}, \frac{(\omega \text{Re})^{\frac{1}{8}}}{(\omega(1-\omega))^{\frac{3}{8}}} \right\}, \\ \kappa_2 &:= \max \left\{ 1, \sqrt{\frac{\text{Re}}{1-\omega}}, \sqrt{\text{We}} \right\}, \\ \kappa_3 &:= \max \left\{ \frac{1}{\sqrt{\omega(1-\omega)}}, \frac{\sqrt{\text{Re}}}{\sqrt{\omega(1-\omega)}}, \frac{(\text{Re})^{\frac{1}{4}}}{\omega^{\frac{1}{2}}(1-\omega)^{\frac{3}{4}}}, \frac{(\text{We})^{\frac{1}{4}}}{\sqrt{\omega(1-\omega)}} \right\}. \end{aligned}$$

Then (3.9) and (3.16) read as follows:

$$(3.17) \quad E_1(t) \leq E_1(0) + C \kappa_1 E(t)^{\frac{3}{2}}, \quad \text{for } -\frac{d}{2} < s < \frac{d}{2} - 1,$$

and

$$(3.18) \quad E_2(t) \leq \kappa_2 E_2(0) + C \kappa_2 \kappa_3 E(t)^2, \quad \text{for } -\frac{d}{2} < s < \frac{d}{2} - 1,$$

Consequently,

$$(3.19) \quad E(t) \leq \kappa_2 E(0) + C \left(\kappa_1 E(t)^{\frac{3}{2}} + \kappa_2 \kappa_3 E(t)^2 \right), \quad \text{for } -\frac{d}{2} < s < \frac{d}{2} - 1.$$

By using standard continuity method, we infer from (3.19) that

$$(3.20) \quad E(t) \leq 2\kappa_2 E(0),$$

provided $E(0)$ is small enough. Then the existence part of Theorem 1.1 follows immediately. \square

4. UNIQUENESS

Let (u_1, τ_1) and (u_2, τ_2) be the solution to the system (1.2) with the same initial data obtained in Section 3. Denote $(w, \sigma) := (u_1 - u_2, \tau_1 - \tau_2)$, and $p = \Pi_1 - \Pi_2$. Then it is easy to verify that (w, σ) satisfies the following system:

$$(4.1) \quad \begin{cases} \text{Re } \partial_t w - (1-\omega) \Delta w + \nabla p = \text{div } \sigma - \text{Re } (w \cdot \nabla u_1 + u_2 \cdot \nabla w), \\ \text{We } (\partial_t \sigma + u_1 \cdot \nabla \sigma) + \sigma = 2\omega D(w) - \text{We } w \cdot \nabla \tau_2 - \text{We } g_\alpha(\sigma, \nabla u_1) - \text{We } g_\alpha(\tau_2, \nabla w). \end{cases}$$

Applying the localized operator $\dot{\Delta}_q$ to system (4.1) yields

$$(4.2) \quad \begin{cases} \text{Re } \partial_t \dot{\Delta}_q w - (1-\omega) \Delta \dot{\Delta}_q w + \nabla \dot{\Delta}_q p = \text{div } \dot{\Delta}_q \sigma - \text{Re } \dot{\Delta}_q (w \cdot \nabla u_1 + u_2 \cdot \nabla w), \\ \text{We } (\partial_t \dot{\Delta}_q \sigma + u_1 \cdot \nabla \dot{\Delta}_q \sigma) + \dot{\Delta}_q \sigma = 2\omega D(\dot{\Delta}_q w) - \text{We } [\dot{\Delta}_q, u_1] \cdot \nabla \sigma - \text{We } \dot{\Delta}_q (w \cdot \nabla \tau_2) \\ \quad - \text{We } \dot{\Delta}_q g_\alpha(\sigma, \nabla u_1) - \text{We } \dot{\Delta}_q g_\alpha(\tau_2, \nabla w). \end{cases}$$

Using the cancelation relation $(\operatorname{div} \dot{\Delta}_q \sigma | \dot{\Delta}_q w) + (D(\dot{\Delta}_q w) | \dot{\Delta}_q \sigma) = 0$, similar to (3.2), we arrive at

$$\begin{aligned}
 & \frac{1}{2} \frac{d}{dt} \left(2\omega \operatorname{Re} \| \dot{\Delta}_q w \|_{L^2}^2 + \operatorname{We} \| \dot{\Delta}_q \sigma \|_{L^2}^2 \right) + 2\omega(1-\omega) \| \nabla \dot{\Delta}_q w \|_{L^2}^2 + \| \dot{\Delta}_q \sigma \|_{L^2}^2 \\
 & \leq 2\omega \operatorname{Re} \| \dot{\Delta}_q (w \cdot \nabla u_1 + u_2 \cdot \nabla w) \|_{L^2} \| \dot{\Delta}_q w \|_{L^2} + \operatorname{We} \left(\| [\dot{\Delta}_q, u_1] \cdot \nabla \sigma \|_{L^2} + \| \dot{\Delta}_q (w \cdot \nabla \tau_2) \|_{L^2} \right) \| \dot{\Delta}_q \sigma \|_{L^2} \\
 (4.3) \quad & + \operatorname{We} \left(\| \dot{\Delta}_q g_\alpha(\sigma, \nabla u_1) \|_{L^2} + \| \dot{\Delta}_q g_\alpha(\tau_2, \nabla w) \|_{L^2} \right) \| \dot{\Delta}_q \sigma \|_{L^2}.
 \end{aligned}$$

Integrating (4.3) w. r. t. time t , multiplying the resulting inequality by 2^{2qs} , and then taking sum w. r. t. q over \mathbb{Z} , using Hölder's inequality, we are led to

$$\begin{aligned}
 & \omega \operatorname{Re} \| w(t) \|_{\dot{H}^s}^2 + \frac{\operatorname{We}}{2} \| \sigma(t) \|_{\dot{H}^s}^2 + 2\omega(1-\omega) \| \nabla w \|_{L_t^2(\dot{H}^s)}^2 + \| \sigma \|_{L_t^2(\dot{H}^s)}^2 \\
 & \leq 2\omega \operatorname{Re} \int_0^t \| w \cdot \nabla u_1 + u_2 \cdot \nabla w \|_{\dot{H}^s} \| w \|_{\dot{H}^s} dt' + \operatorname{We} \int_0^t \left(\sum_{q \in \mathbb{Z}} 2^{2qs} \| [\dot{\Delta}_q, u_1] \cdot \nabla \sigma \|_{L^2}^2 \right)^{\frac{1}{2}} \| \sigma \|_{\dot{H}^s} dt' \\
 (4.4) \quad & + \operatorname{We} \int_0^t \left(\| w \cdot \nabla \tau_2 \|_{\dot{H}^s} + \| g_\alpha(\sigma, \nabla u_1) \|_{\dot{H}^s} + \| g_\alpha(\tau_2, \nabla w) \|_{\dot{H}^s} \right) \| \sigma \|_{\dot{H}^s} dt'.
 \end{aligned}$$

By virtue of the product estimates (2.4), (2.5) and commutator estimate in Besov spaces, we have

$$\begin{aligned}
 & \| w \cdot \nabla u_1 + u_2 \cdot \nabla w \|_{\dot{H}^s} \lesssim \| \nabla u_1 \|_{\dot{B}_{2,1}^{\frac{d}{2}}} \| w \|_{\dot{H}^s} + \| u_2 \|_{\dot{B}_{2,1}^{\frac{d}{2}}} \| \nabla w \|_{\dot{H}^s}, \quad \text{for } -\frac{d}{2} < s < \frac{d}{2}, \\
 & \left(\sum_{q \in \mathbb{Z}} 2^{2qs} \| [\dot{\Delta}_q, u_1] \cdot \nabla \sigma \|_{L^2}^2 \right)^{\frac{1}{2}} \lesssim \| \nabla u_1 \|_{\dot{B}_{2,1}^{\frac{d}{2}}} \| \sigma \|_{\dot{H}^s}, \quad \text{for } -\frac{d}{2} - 1 < s < \frac{d}{2}, \\
 & \| w \cdot \nabla \tau_2 \|_{\dot{H}^s} \lesssim \| w \|_{\dot{H}^{s+1}} \| \nabla \tau_2 \|_{\dot{B}_{2,\infty}^{\frac{d}{2}-1}} \lesssim \| \nabla w \|_{\dot{H}^s} \| \tau_2 \|_{\dot{B}_{2,1}^{\frac{d}{2}}}, \quad \text{for } -\frac{d}{2} < s < \frac{d}{2} - 1, \\
 & \| g_\alpha(\sigma, \nabla u_1) \|_{\dot{H}^s} + \| g_\alpha(\tau_2, \nabla w) \|_{\dot{H}^s} \lesssim \| \nabla u_1 \|_{\dot{B}_{2,1}^{\frac{d}{2}}} \| \sigma \|_{\dot{H}^s} + \| \tau_2 \|_{\dot{B}_{2,1}^{\frac{d}{2}}} \| \nabla w \|_{\dot{H}^s}, \quad \text{for } -\frac{d}{2} < s < \frac{d}{2},
 \end{aligned}$$

Substituting these estimates into (4.4) yields that, for $-\frac{d}{2} < s < \frac{d}{2} - 1$, there holds

$$\begin{aligned}
 & \omega \operatorname{Re} \| w(t) \|_{\dot{H}^s}^2 + \operatorname{We} \| \sigma(t) \|_{\dot{H}^s}^2 + 2\omega(1-\omega) \| \nabla w \|_{L_t^2(\dot{H}^s)}^2 + \| \sigma \|_{L_t^2(\dot{H}^s)}^2 \\
 & \leq C \int_0^t \| \nabla u_1 \|_{\dot{B}_{2,1}^{\frac{d}{2}}}^{\frac{d}{2}} \left(\omega \operatorname{Re} \| w \|_{\dot{H}^s}^2 + \operatorname{We} \| \sigma \|_{\dot{H}^s}^2 \right) dt' \\
 (4.5) \quad & + C \int_0^t \left(\omega \operatorname{Re} \| u_2 \|_{\dot{B}_{2,1}^{\frac{d}{2}}}^{\frac{d}{2}} \| w \|_{\dot{H}^s} + \operatorname{We} \| \tau_2 \|_{\dot{B}_{2,1}^{\frac{d}{2}}}^{\frac{d}{2}} \| \sigma \|_{\dot{H}^s} \right) \| \nabla w \|_{\dot{H}^s} dt'.
 \end{aligned}$$

Noting that by Cauchy's inequality, there exists a positive constant C depending on Re , We and ω , such that

$$\begin{aligned}
 & \int_0^t \left(\omega \operatorname{Re} \| u_2 \|_{\dot{B}_{2,1}^{\frac{d}{2}}}^{\frac{d}{2}} \| w \|_{\dot{H}^s} + \operatorname{We} \| \tau_2 \|_{\dot{B}_{2,1}^{\frac{d}{2}}}^{\frac{d}{2}} \| \sigma \|_{\dot{H}^s} \right) \| \nabla w \|_{\dot{H}^s} dt' \\
 & \leq \omega(1-\omega) \| \nabla w \|_{L_t^2(\dot{H}^s)}^2 + C \int_0^t \left(\omega \operatorname{Re} \| u_2 \|_{\dot{B}_{2,1}^{\frac{d}{2}}}^2 + \operatorname{We} \| \tau_2 \|_{\dot{B}_{2,1}^{\frac{d}{2}}}^2 \right) \left(\omega \operatorname{Re} \| w \|_{\dot{H}^s}^2 + \operatorname{We} \| \sigma \|_{\dot{H}^s}^2 \right) dt',
 \end{aligned}$$

combining this inequality with (4.5), we obtain

$$\begin{aligned}
 & \omega \operatorname{Re} \| w(t) \|_{\dot{H}^s}^2 + \operatorname{We} \| \sigma(t) \|_{\dot{H}^s}^2 + \omega(1-\omega) \| \nabla w \|_{L_t^2(\dot{H}^s)}^2 + \| \sigma \|_{L_t^2(\dot{H}^s)}^2 \\
 & \leq C \int_0^t \left(\| \nabla u_1 \|_{\dot{B}_{2,1}^{\frac{d}{2}}}^{\frac{d}{2}} + \omega \operatorname{Re} \| u_2 \|_{\dot{B}_{2,1}^{\frac{d}{2}}}^2 + \operatorname{We} \| \tau_2 \|_{\dot{B}_{2,1}^{\frac{d}{2}}}^2 \right) \left(\omega \operatorname{Re} \| w \|_{\dot{H}^s}^2 + \operatorname{We} \| \sigma \|_{\dot{H}^s}^2 \right) dt'.
 \end{aligned}$$

Thanks to the embedding in low frequency, we infer from (3.20) that

$$\begin{aligned}
& \int_0^t \left(\|\nabla u_1\|_{\dot{B}_{2,1}^{\frac{d}{2}}}^2 + \omega \operatorname{Re} \|u_2\|_{\dot{B}_{2,1}^{\frac{d}{2}}}^2 + \operatorname{We} \|\tau_2\|_{\dot{B}_{2,1}^{\frac{d}{2}}}^2 \right) dt' \\
& \lesssim \|\nabla u_1\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}})}^h + t^{\frac{1}{2}} \|\nabla u_1\|_{L_t^2(\dot{H}^s)} + \omega \operatorname{Re} t \left(\|u_2\|_{\tilde{L}_t^\infty(\dot{B}_{2,1}^{\frac{d}{2}})}^h + \|u_2\|_{\tilde{L}_t^\infty(\dot{H}^s)} \right)^2 \\
& \quad + \operatorname{We} t \left(\|\tau_2\|_{\tilde{L}_t^\infty(\dot{B}_{2,1}^{\frac{d}{2}})}^h + \|\tau_2\|_{\tilde{L}_t^\infty(\dot{H}^s)} \right)^2 \\
& \leq \infty,
\end{aligned}$$

for any $t > 0$. Then the uniqueness follows from Gronwall's inequality immediately. This completes the proof of Theorem 1.1. \square

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REFERENCES

- [1] H. Bahouri, J.-Y. Chemin and R. Danchin, “Fourier analysis and nonlinear partial differential equations”. Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], 343. Springer, Heidelberg, 2011.
- [2] J.-M. Bony, Calcul symbolique et propagation des singularités pour équations aux dérivées partielles nonlinéaires. *Annales Scientifiques de l'école Normale Supérieure* **14**(1981), 209–246.
- [3] J.-Y. Chemin, “Localization in Fourier space and Navier-Stokes system, Phase Space Analysis of Partial Differential Equations”. Proceedings 2004, CRM series, Pisa 2004, 53–136.
- [4] J.-Y. Chemin, N. Lerner Flot de champs de vecteurs non Lipschitziens et équations de Navier-Stokes. *J. Differential Equations*, **121**(1995), 314–228.
- [5] J. Y. Chemin, N. Masmoudi, About lifespan of regular solutions of equations related to viscoelastic fluids. *SIAM J. Math. Anal.*, **33**(2001), 84–112.
- [6] Q. L. Chen, C. X. Miao, Global well-posedness of viscoelastic fluids of Oldroyd type in Besov spaces, *Nonlinear Analysis*, **68**(2008), 1928–1939.
- [7] P. Constantin, M. Kliegl, Note on global regularity for two-dimensional Oldroyd-B fluids with diffusive stress. *Arch. Rational Mech. Anal.*, **206**(2012), 725–740.
- [8] D.Y. Fang, M. Hieber and R. Z. Zi, Global existence results for Oldroyd-B Fluids in exterior domains: The case of non-small coupling parameters, *Math. Ann.*, **357**(2013), 687–709.
- [9] D. Y. Fang, R. Z. Zi, Incompressible limit of Oldroyd-B fluids in the whole space. *J. Differential Equations*, **256**(2014), 2559–2602.
- [10] E. Fernández-Cara, F. Guillén, R. Ortega, Some theoretical results concerning non-Newtonian fluids of the Oldroyd kind. *Ann. Scuola Norm. Sup. Pisa*, **26**(1998), 1–29.
- [11] C. Guillopé, J. C. Saut, Existence results for the flow of viscoelastic fluids with a differential constitutive law. *Non-linear Anal.* **15**(1990), 849–869.
- [12] C. Guillopé, Z. Salloum and R. Talhouk, Regular flows of weakly compressible viscoelastic fluids and the incompressible limit. *Discrete Contin. Dyn. Syst. Ser. B* **14**(2010), 1001–1028.
- [13] M. Hieber, Y. Naito and Y. Shibata, Global existence results for Oldroyd-B fluids in exterior domains. *J. Differential Equations*, **252** (2012) 2617–2629.
- [14] X. P. Hu, F. H. Lin, Global solutions of two dimensional incompressible viscoelastic flows with discontinuous initial data, arXiv:1312.6749 [math.AP].
- [15] R. Kupferman, C. Mangoubi, and E. S. Titi, A Beale-Kato-Majda breakdown criterion for an Oldroyd-B fluid in the creeping flow regime. *Commun. Math. Sci.*, **6**(2008), 235–256.
- [16] Z. Lei, Global existence of classical solutions for some Oldroyd-B model via the incompressible limit. *Chinese Ann. Math. Ser. B*, **27** (2006), 565–580.
- [17] Z. Lei, On 2D viscoelasticity with small strain, *Arch. Ration. Mech. Anal.*, **198**(2010), 13–37.
- [18] Z. Lei, C. Liu and Y. Zhou, Global existence for a 2D incompressible viscoelastic model with small strain. *Commun. Math. Sci.*, **5**(2007), 595–616.
- [19] Z. Lei, C. Liu and Y. Zhou, Global solutions for incompressible viscoelastic fluids, *Arch. Ration. Mech. Anal.*, **188**(2008), 371–398.

- [20] Z. Lei, N. Masmoudi, Y. Zhou, Remarks on the blowup criteria for Oldroyd models, *J. Differential Equations*, **248**(2010), 328–341.
- [21] Z. Lei, Y. Zhou, Global existence of classical solutions for the two-dimensional Oldroyd model via the incompressible limit, *SIAM J. Math. Anal.*, **37**(2005), 797–814.
- [22] F. H. Lin, C. Liu and P. Zhang, On hydrodynamics of viscoelastic fluids, *Comm. Pure Appl. Math.*, **58**(2005), 1437–1471.
- [23] F. H. Lin, P. Zhang, On the initial-boundary value problem of the incompressible viscoelastic fluid system, *Comm. Pure Appl. Math.*, **61**(2008), 539–558.
- [24] P. L. Lions, N. Masmoudi, Global solutions for some Oldroyd models of non-Newtonian flows. *Chinese Ann. Math. Ser. B*, **21**(2000), 131–146.
- [25] L. Molinet, R. Talhouk, On the global and periodic regular flows of viscoelastic fluids with a differential constitutive law. *Nonlinear Diff. Equations Appl.* **11**(2004), 349–359.
- [26] J. G. Oldroyd, Non-Newtonian effects in steady motion of some idealized elastico-viscous liquids. *Proc. Roy. Soc. London.*, **245**(1958), 278–297.
- [27] J. Z. Qian, Well-posedness in critical spaces for incompressible viscoelastic fluid system, *Nonlinear Analysis*, **72**(2010), 3222–3234.
- [28] R. Talhouk, “Analyse Mathématique de Quelques Écoulements de Fluides Viscoélastiques,” Thèse, Université Paris-Sud, 1994.
- [29] T. Zhang, D. Y. Fang, Global existence of strong solution for equations related to the incompressible viscoelastic fluids in the critical L^p framework, *SIAM J. Math. Anal.*, **44**(2012), 2266–2288.
- [30] R. Z. Zi, D. Y. Fang and T. Zhang, Global solution to the incompressible Oldroyd-B model in the critical L^p framework: the case of the non-small coupling parameter, *Arch. Rational Mech. Anal.*, **213**(2014), 651–687.

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